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Abstract
The interest in decoherence-free, or noiseless subsystems (DFS/NSs) of quantum systems is both of fundamental and of practical interest. Understanding the invariance of a set of states under certain transformations is mutually associated with a better understanding of some fundamental aspects of quantum mechanics as well as the practical utility of invariant subsystems. For example, DFS/NSs are potentially useful for protecting quantum information in quantum cryptography and quantum computing as well as enabling universal computation. Here we discuss transformations which are compatible with a DFS/NS that is composed of d-state systems which protect against collective noise. They are compatible in the sense that they do not take the logical (encoded) states outside of the DFS/NS during the transformation. Furthermore, it is shown that the Hamiltonian evolutions derived here can be used to perform universal quantum computation on a three qudit DFS/NS. Many of the methods used in our derivations are directly applicable to a large variety of DFS/NSs. More generally, we may also state that these transformations are compatible with collective motions.

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1. Introduction

In 1954, Dicke argued that an independent approximation for radiating physical systems was very often not reasonable [1]. Many physical systems are correlated, and transform in a similar, or even identical way so that an independent approximation should not be used. For some systems the motions can be considered collective. For example, this can be the case when a set of particles interacts with a common field. In proposed quantum computing devices, interactions with a common field can lead to collective noise which is an *unwanted* interaction.

This collective noise, or collective decoherence, will cause quantum information to be lost to the environment [2, 3].

To avoid the detrimental effects of such noise, a theory of decoherence-free or noiseless subsystems (DFS/NSs) was formulated [4–9]. (For reviews see [10, 11].) When information is encoded into one particular type of DFS/NS it is protected against collective errors. More generally, under suitable conditions a DFS/NS can also protect against noises which are not of the collective type, but still correspond to an identifiable symmetry in the system–bath interaction. However, in this paper we will focus on collective DFS/NSs. Once information is encoded into a DFS/NS, it must then be manipulated if the quantum information is to be used for the purposes of computation or simulation. Not just any physically available operation is acceptable. The manipulations, or gating operations, must be *compatible* with the DFS/NS [8] if the information is to remain protected. Here compatible means that the operations should not take the information outside of the DFS/NS. For, if they did, the information would be vulnerable to collective noises during the time it is not confined to the DFS/NS.

The objective of this paper is to describe Hamiltonian evolutions which are compatible with collective operations on the system in the sense that they commute with the collective operations. The Hamiltonians we find here will provide a set of transformations which are compatible with a DFS/NS that protects information from collective noise and is composed of qudits.

Qudits are quite interesting systems for several reasons. For example, two three-state systems, or qutrits, can be more entangled than two qubits [12–14]. There is also evidence suggesting that a collection of d-state systems can share a larger pairwise fraction of their entanglement capacity as the dimension increases [15]. They are beneficial for several information processing tasks including cryptography [16–20], computing [21–23], and games. They seem to be *required* for a version of the Byzantine agreement problem [24]. So for these reasons, among others, universality requirements for qudits [25–30] as well as several recent experiments and proposed experiments have provided methods for producing and manipulating qudits [31–34] and an analysis of the collective behavior for an assembly of qutrits has been given [35].

Collective DFS/NSs have been studied quite thoroughly for a variety of reasons. One is that collective errors are physically observed in some systems. Another is that a collective DFS encoding can enable universal computation on a set of qubits [8, 26, 36–44]. Collective DFS/NSs have also been observed to reduce noise in several experiments [45–49], including computation in a qubit DFS [50, 51]. Even if collective errors are not present in an experiment, it is possible to induce such a symmetry using decoupling operations [52–57]. Theoretically collective noises are easier to treat in part because a basis for collective operations forms a representation of the algebra of the special unitary group [58] acting on the constituents. This enables a variety of group-theoretical methods to be employed in their treatment. We will see here that this is also the case for computing in a qudit DFS/NS.

Specifically in this paper, section 2 contains a review of DFS/NS theory. We then provide a description of the compatibility condition and show how it can be used to find a set of Hamiltonians which enable compatible computation on a DFS/NS in section 3. Section 4 contains a set of compatible Hamiltonians for a system of three qudits. We provide a complete set for qutrits in section 4.1 and then generalize to qudits. In section 5 we show how to analytically obtain several unitary transformations on encoded states using these results. Section 6 extends this analysis to a system of n-qudits ($n \geqslant 3$), thereby showing that these compatible Hamiltonians can be used to manipulate encoded quDit states ($D \neq d$) of arbitrary dimension. We conclude in section 7 with a discussion of our results and their implication

for universality using a three qudit DFS/NS. Two appendices have also been included which contain some detailed calculations.

2. Decoherence-free/noise less subsystems

Here a brief review of DFS/NSs is provided. Although the description is, by now, more or less standard, we will primarily follow the notation of [8]. For further details, see [7] and/or [8]. We note that one may describe a DFS/NS in terms of a collection of operators appearing in a semi-group master equation [6]. We will, however, use the Hamiltonian description here.

2.1. Subsystem structure

Consider a Hamiltonian which includes a term describing the evolution of the system (H_S), a term describing the evolution of a bath or environment (H_B), and an interaction term describing the evolution of the system and environment together (H_{SB}),

$$H = H_S + H_B + H_{SB}. \tag{1}$$

Without loss of generality, we will write the interaction term as

$$H_{SB} = \sum_\alpha S_\alpha \otimes B_\alpha, \tag{2}$$

where the S_α act only on the system and the B_α act only on the bath. Let us now consider the algebra $\mathcal{A}$ generated by the set of operators $\{H_S, S_\alpha\}$. We will assume that this algebra is reducible and thus the operators can all be simultaneously block diagonalized with a single unitary transformation. Such algebras are said to be, mathematically, completely reducible, meaning they may be completely block diagonalized, i.e., written as a direct sum of irreducible components. It is this reducibility, which follows for any system–bath interaction symmetry that is preserved in the algebra, that allows for a non-trivial decomposition of the algebra into regions supporting the logical encoding of protected information. It is noteworthy that the 'unitary trick' can be used to relate algebraic representation theory and group representation theory [58]. This is particularly useful for collective errors acting on systems of qudits since these operations form a representation of the algebra of such systems.

   The aforementioned decomposition is described by the equation

$$\mathcal{A} \cong \bigoplus_{J \in \mathcal{J}} \mathbb{1}_{n_J} \otimes \mathcal{M}(d_J, \mathbb{C}), \tag{3}$$

where the n_J-fold degenerate $d_J \times d_J$ complex matrices $\mathcal{M}(d_J, \mathbb{C})$ correspond to the irreducible components of $\mathcal{A}$. We label these components by J, which collectively form the finite set $\mathcal{J}$. (It is important to note that this J actually stands for a set of quantum numbers when the constituents are qudits, with $d \geqslant 3$.) One may also define the commutant which is the set of elements that commute with all elements of the algebra $\mathcal{A}$. The form of these is dictated by the irreducibility of the blocks in $\mathcal{A}$,

$$\mathcal{A}' = \{X : [X, A] = 0, \forall A \in \mathcal{A}\}. \tag{4}$$

To equation (3) there is a corresponding decomposition of the Hilbert space $\mathcal{H}_S = \sum_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}$ where the second factor corresponds to the part of the Hilbert space which is affected by noise ($\mathbb{C}^{d_J}$) and the first factor corresponds to the part which is not ($\mathbb{C}^{n_J}$).

   The unitary transformation which is used to change between the physical and logical bases can be referred to as the DFS/NS transformation. In the logical basis, the elements of $\mathcal{A}$ exhibit a structure that allows quantum information to remain confined to the logical subspaces

while the physical system interacts with its environment. The superpositions of physical states which form the logical, or encoded, states of the DFS/NS are created by this transformation in the following way. Let V_{dfs} be the aforementioned transformation that simultaneously block diagonalizes each element of the algebra $A_i \in \mathcal{A}$ in an identical way, i.e., $A_i' = V_{\text{dfs}} A_i V_{\text{dfs}}^{-1}$. The physical states $|\Psi_p\rangle$ are then related to the logical states $|\Psi_L\rangle$ by

$$|\Psi_L\rangle = V_{\text{dfs}} |\Psi_p\rangle. \tag{5}$$

In practice, V_{dfs} can be used to express states and operators in the physical bases in terms of states and operators in the logical basis as done explicitly, for example, in [62].

We may now define a decoherence-free or noiseless *subsystem* in the following way. Suppose that we represent a basis of eigenstates corresponding to a particular J with the set $\{|\lambda\rangle \otimes |\mu\rangle\}$, where $\lambda = 1, \ldots, n_J$ and $\mu = 1, \ldots, d_J$. Then if

$$A_\alpha |\lambda\rangle \otimes |\mu\rangle = \sum_{\mu'=1}^{d_J} M_{\mu\mu',\alpha} |\lambda\rangle \otimes |\mu'\rangle \tag{6}$$

for all A_α, λ and μ, there exists an irreducible decomposition as given by equation (3). The invariance of the degeneracy labels $|\lambda\rangle$ in this last expression reflects the ability to reliably store quantum information in certain regions of the system Hilbert space when the algebra $\mathcal{A}$ can be decomposed in this way. The information is stored in blocks with the same J but different λ. Each λ specifies a particular DFS/NS basis state. These logical states can therefore be expanded in terms of those states associated with a given λ. Although the initial encoding of a particular logical state may change, it will remain confined to its initial subspace.

A decoherence-free *subspace* is one for which the matrices M are numbers (1×1 matrices) which act on a one-dimensional representation, i.e., a singlet state.

2.2. DFS and NS examples

In this section we review a few examples of DFS/NSs which will aid in our discussion of qudit systems. The examples will be useful for comparing and contrasting certain properties of qubit and qudit systems.

2.2.1. Four-qubit DFS. As stated above a decoherence-free subspace is comprised of singlet states. Four qubits can be used to construct a DFS/NS qubit which is represented by two singlet states, one singlet state for the logical zero and one for the logical one [6]. The logical states considered here protect the information from errors which act the same on each of the four physical qubits constituting the system. These states are given explicitly by

$$|0_L\rangle = (|0101\rangle + |1010\rangle - |0110\rangle - |1001\rangle)/2 \tag{7}$$

and

$$|1_L\rangle = (2|0011\rangle + 2|1100\rangle - |0110\rangle - |1001\rangle - |0101\rangle - |1010\rangle)/\sqrt{12}, \tag{8}$$

where $|0\rangle$ and $|1\rangle$ represent two orthogonal basis states for a spin-1/2 particle.

It turns out that the Heisenberg exchange interaction is universal for a set of qubits constructed in this way and logical gates consisting of only these interactions have been provided [8, 36, 37, 59]. (In this paper we neglect the corrections which must be made to the logical gates in order to account for three- and four-body interaction terms. For a discussion of these effects see, for example, [60].) The Heisenberg exchange interaction between pairs of physical qubits can be expressed as

$$E_{ij} = \tfrac{1}{2}(I + \vec{\sigma}_i \cdot \vec{\sigma}_j), \tag{9}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. As written E_{ij} is the exchange operation between qubits i and j, i.e., $E_{ij}|\phi\rangle_i|\psi\rangle_j = |\psi\rangle_i|\phi\rangle_j$. The logical '$X$' operation is given by

$$\bar{X} = \tfrac{1}{\sqrt{3}}(E_{23} - E_{13}). \tag{10}$$

The logical 'Z' operation is given by

$$\bar{Z} = -E_{12} \tag{11}$$

and $\bar{Y}$ can be obtained from these two by commutation or the finite transformations can be written in terms of Euler angles.

2.2.2. Three-qubit DFS. The smallest number of qubits that enables the DFS/NS encoding of a logical qubit is three. In this case the logical zero and logical one are each represented by a doublet. In the $\{|0\rangle, |1\rangle\}$ basis, these states can be written as

$$|0_L\rangle = \alpha_0(|010\rangle - |100\rangle)/\sqrt{2} + \beta_0(|011\rangle - |101\rangle)/\sqrt{2}, \tag{12}$$

and

$$|1_L\rangle = \alpha_1(2|001\rangle - |010\rangle - |100\rangle)/\sqrt{6} + \beta_1(-2|110\rangle + |011\rangle + |101\rangle)/\sqrt{6}, \tag{13}$$

with $|\alpha_i|^2 + |\beta_i|^2 = 1$. As in the previous example, these logical states protect the quantum information from collective errors. Although the Heisenberg exchange interaction is universal for both the three-qubit and four-qubit DFS/NSs, there are some practical differences between the respective logical operations for these systems with regard to error prevention [61, 62]. We will not discuss this here, but will note the form of the logical operations. The logical 'X' operation is given by [8]

$$\bar{X} = \tfrac{1}{\sqrt{3}}(E_{23} - E_{13}), \tag{14}$$

while the logical 'Z' operation is given by[3]

$$\bar{Z} = \tfrac{1}{3}(E_{13} + E_{23} - 2E_{12}). \tag{15}$$

Again, $\bar{Y}$ can be obtained from these two by commutation or Euler angles.

It is important to note the similarities and differences here. However, for our purposes, the most important point is that the Heisenberg exchange interaction can be used to construct a universal set of operations for both the three- and four-qubit DFS.

2.2.3. Three-qutrit NS. Here we provide some details concerning the logical qubit encoding over a subspace of three physical qutrits. The purpose of this particular example is to provide enough structure to enable a smooth transition into our discussion of qudit systems.

The three qutrit DFS/NS is quite analogous to the three qubit DFS/NS discussed above in many respects although the dimensions of the subspaces are different. For the three qubit NS, a tensor product of three qubits can be decomposed into two two-state subsystems and a four-state system. This may be written as $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4}$. A tensor product of three qutrits can be decomposed into a singlet, a decuplet and two octets:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \oplus \mathbf{10}, \tag{16}$$

where the **8** identifies an eight-dimensional representation (octet), the **1** identifies a one-dimensional representation (singlet), etc. A logical qubit can be represented by two degenerate

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[3] We note that the corresponding equation (74) in [8] contains incorrect numerical factors, which we have fixed here.

eight-state subsystems, the two octets [58]. We may also note that it is possible to experimentally produce such states encoded in a set of polarized photons [32].

Explicit forms for the states within the two octet subspaces of three physical qutrits were provided in [58]. The three states within each individual qutrit will be denoted $|0\rangle$, $|1\rangle$ or $|2\rangle$. In terms of the quantum numbers p, q, t, t_3 and y as given in [58], these are[4]

$$
\begin{aligned}
|0\rangle &= |0, 1, -1/2, -1/2, -1/3\rangle, \\
|1\rangle &= |0, 1, -1/2, 1/2, -1/3\rangle, \\
|2\rangle &= |0, 1, 0, 0, 2/3\rangle.
\end{aligned}
\tag{17}
$$

The physical, or computational basis ($|000\rangle, |001\rangle, \ldots, |222\rangle$) transforms into the DFS basis via the operation V_{dfs} which is the matrix composed of the Wigner–Clebsh–Gordan coefficients. In this noiseless basis the logical zero state $|0_L\rangle$ is formed from the octet which corresponds to the states [58]

$$
\begin{aligned}
\psi_1^{8,0} &= (|200\rangle - |020\rangle)/\sqrt{2}, \\
\psi_2^{8,0} &= (|100\rangle - |010\rangle)/\sqrt{2}, \\
\psi_3^{8,0} &= (|011\rangle - |101\rangle)/\sqrt{2}, \\
\psi_4^{8,0} &= (|211\rangle - |121\rangle)/\sqrt{2}, \\
\psi_5^{8,0} &= (|122\rangle - |212\rangle)/\sqrt{2}, \\
\psi_6^{8,0} &= (|022\rangle - |202\rangle)/\sqrt{2}, \\
\psi_7^{8,0} &= (-|021\rangle - |120\rangle + |201\rangle + |210\rangle)/2, \\
\psi_8^{8,0} &= (2|012\rangle + |021\rangle - 2|102\rangle - |120\rangle - |201\rangle + |210\rangle)/\sqrt{12},
\end{aligned}
\tag{18}
$$

where the first superscript on each ψ denotes the dimension of the representation, the second is a degeneracy label and the subscript labels the state within the representation. A second octet forms the logical one state $|1_L\rangle$,

$$
\begin{aligned}
\psi_1^{8,1} &= (-2|002\rangle + |020\rangle + |200\rangle)/\sqrt{6}, \\
\psi_2^{8,1} &= (-2|001\rangle + |010\rangle + |100\rangle)/\sqrt{6}, \\
\psi_3^{8,1} &= (-2|110\rangle + |011\rangle + |101\rangle)/\sqrt{6}, \\
\psi_4^{8,1} &= (-2|112\rangle + |121\rangle + |211\rangle)/\sqrt{6}, \\
\psi_5^{8,1} &= (-2|221\rangle + |122\rangle + |212\rangle)/\sqrt{6}, \\
\psi_6^{8,1} &= (-2|220\rangle + |022\rangle + |202\rangle)/\sqrt{6}, \\
\psi_7^{8,1} &= (-2|012\rangle + |021\rangle - 2|102\rangle + |120\rangle + |201\rangle + |210\rangle)/\sqrt{12}, \\
\psi_8^{8,1} &= (|021\rangle - |120\rangle + |201\rangle - |210\rangle)/2.
\end{aligned}
\tag{19}
$$

In terms of these octets the logical zero state is given by an arbitrary superposition of the eight $\psi_j^{8,0}$ states, $|0_L\rangle = \sum_j \alpha_j \psi_j^{8,0}$ and likewise for $|1_L\rangle = \sum_j \beta_j \psi_j^{8,1}$. According to the theory of noiseless subsystems, states within the sets $\{\psi_i^{8,0}\}$ ($\{\psi_i^{8,1}\}$) will mix together in the presence of collective noise but not with the states in $\{\psi_i^{8,1}\}$ ($\{\psi_i^{8,0}\}$) or with the one- or ten-dimensional representations. Furthermore, the mixing will be identical for both

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[4] Note that the basis being used here is for the set of Gell–Mann matrices. For various physical systems, a basis is chosen which corresponds to the set of 'good quantum numbers' for the particular system.

octets. This means that when only collective errors are present, the information encoded in $|\psi_L\rangle = a|0_L\rangle + b|1_L\rangle$ will be protected. It is also important to note that the initialization is arbitrary in the sense that an arbitrary combination can be taken [63]. In practical situations this can be very beneficial. (See for example [32].)

2.2.4. Three-qudit DFS. As discussed in [58] the tensor product of three equivalent irreducible representations of $SU(d)$ gives rise to the smallest number of qudits for which a NS, representing a qubit in terms of qudits, exists. This can be seen in the tensor product of three d-state systems using the tableau of a representation. Taking the tensor product of three such systems produces two degenerate tableau. These correspond to two degenerate irreducible representations which can be used to store protected quantum information in the form of a noiseless qubit.

As we will see later, this decomposition is valid for all three-qudit systems and enables us to calculate the compatible transformations governing the noiseless evolution of such systems.

3. Computing in a DFS/NS

When information is encoded in a DFS/NS, it will be protected during the unitary transformations which are used to manipulate it only if those transformations do not couple states inside the subspace with states outside of the subspace. In other words, we must restrict the transformations to those which preserve the subspace structure. When this can be accomplished, we say that such logical operations are *compatible* with the DFS/NS. In order to find a DFS-compatible set of gates we will find it useful to first identify the stabilizer of a DFS.

3.1. The stabilizer of a DFS/NS

Let $|\Psi\rangle \in \mathcal{C}$ be a state in the DFS, where $\mathcal{C}$ denotes the code space. Let us define a stabilizer, $\mathcal{S}$, as in [8]:

$$\mathcal{S} = \{S : S|\Psi\rangle = |\Psi\rangle, \forall |\Psi\rangle \in \mathcal{C}\}.$$

While the elements of this set do in fact leave all of the code words unchanged, we can relax this requirement slightly when considering the DFS states since in this case the information being encoded is stored in the states labeled by λ. This property of the DFS/NS states allows us to define a modified version of the stabilizer above as

$$\mathcal{S}' = \left\{ S' : S'(|\lambda\rangle \otimes |\mu\rangle) = |\lambda\rangle \otimes \sum_{\mu'=1}^{d_J} C_{\mu\mu'}|\mu'\rangle, \forall |\Psi\rangle \in \mathcal{C} \right\}.$$

We can parametrize elements S' of the modified stabilizer which are relevant to the system using equation (6) and a set of arbitrary complex numbers $\{v_\alpha\}$:

$$D(v_1, v_2, \ldots) = \exp\left[\sum_\alpha v_\alpha A_\alpha \right] \tag{20}$$

This leaves the labels λ unchanged, as seen by expanding the exponential and acting term by term on states of the form $|\lambda\rangle \otimes |\mu\rangle$.

3.2. Compatibility conditions for DFS/NS evolution

Using the modified stabilizer, we can state that U is compatible with $\mathcal{C}$, if $\forall |\Psi\rangle \in \mathcal{C}$, $U(|\lambda\rangle \otimes |\mu\rangle) = \sum_{\lambda'} J_{\lambda\lambda'}|\lambda'\rangle \otimes \sum_{\mu'} K_{\mu\mu'}|\mu'\rangle$ and therefore $S'U(|\lambda\rangle \otimes |\mu\rangle) = S'\left(\sum_{\lambda'} J_{\lambda\lambda'}|\lambda'\rangle \otimes \sum_{\mu'} K_{\mu\mu'}|\mu'\rangle\right) = \left(\sum_{\lambda'} J_{\lambda\lambda'}|\lambda'\rangle \otimes \sum_{\mu'} K'_{\mu\mu'}|\mu'\rangle\right) \forall S' \in \mathcal{S}'$. This implies that $U^{-1}S'U(|\lambda\rangle \otimes |\mu\rangle) = |\lambda\rangle \otimes \sum_{\mu'} K''_{\mu\mu'}|\mu'\rangle$ so that

$$U^{-1}S'U \in \mathcal{S}'. \tag{21}$$

The condition equation (21) may now be re-expressed as

$$UD(v_1, v_2, \ldots)U^{\dagger} = S', \tag{22}$$

for some $S' \in \mathcal{S}'$, or

$$U \exp\left[\sum_{\alpha} v_{\alpha} A_{\alpha}\right] U^{\dagger} = \exp\left[\sum_{\alpha} v_{\alpha} U A_{\alpha} U^{\dagger}\right] = S'. \tag{23}$$

Taking the natural logarithm of this equation gives

$$\sum_{\alpha} v_{\alpha} U A_{\alpha} U^{\dagger} = \ln(S'). \tag{24}$$

Now let us define $U = \exp[-iHt]$. Then, after taking the derivative of both sides of this last expression with respect to time, we find a very simple and sufficient, but not necessary condition for the compatibility of Hamiltonians

$$[H, A_{\alpha}] = 0, \qquad \forall A_{\alpha} \in \mathcal{S}'. \tag{25}$$

Then $U = \exp(-iHt)$ is compatible with the DFS/NS. Clearly if the Hamiltonian which generates the unitary U commutes with every element of the algebra, then U also commutes with every element of the algebra. This provides a subset of the unitary transformations U satisfying equation (21).

4. Explicit forms for compatible hamiltonians

It was shown in the previous section how a parametrization of the stabilizer in terms of time independent coefficients v_{α} can be used to identify a set of unitary transformations that are compatible with the DFS/NS states. This section provides a set of those Hamiltonians satisfying equation (25) for systems composed of three qudits. In the following analysis we neglect the internal Hamiltonian of the system and assume that the only interaction terms present are those which can be considered as collective. The set presented here is valid for all $d \geqslant 3$ and it is unique for the case of $d = 3$. The algorithm for determining this set could be used for any DFS/NS, not only those which protect against collective errors.

4.1. Explicit forms for qutrits

Using MATHEMATICA, we have determined the complete set of Hamiltonians which are compatible with an encoded two-state noiseless subsystem of three qutrits (compatible in the sense that they satisfy equation (25)). To do this, we expanded the relevant quantities in a complete set of traceless, Hermitian matrices $\{\lambda_i\}$. For these calculations, we employed the Gell–Mann matrices, but in principle any basis can be used. Several properties and conventions of these matrices are given in appendix A.

Collective errors will be denoted S_{α}, of which there are eight. For example, $S_1 = \sum_i \lambda_1^{(i)}$, where the superscript labels a particular qutrit and the subscript denotes the type of error

labeled 1 to 8. Let $\mu_{ijk} = \lambda_i \otimes \lambda_j \otimes \lambda_k$, where the λ_j are the Gell–Mann matrices with $\lambda_0 = \mathbb{1}$. An arbitrary Hamiltonian can be expanded as $H = \sum_{ijk} a_{ijk}\mu_{ijk}$ with H traceless if not all i, j, k are simultaneously zero and the a_{ijk} are arbitrary real constants. The algorithm we used to find the H which commutes with every collective error proceeds as follows. (We emphasize that this algorithm is quite general and could be used with the elements of any stabilizer to find compatible Hamiltonians.)

(1) Expand H in terms of the complete set of Hermitian matrices μ_{ijk} described above: $H = \sum_{ijk} a_{ijk}\mu_{ijk}$.
(2) Determine the commutator of the general Hamiltonian H and a generic collective error $\sum_i g_i S_i$, where the g_i are arbitrary coefficients. In other words, calculate $\left[H, \sum_i g_i S_i\right]$.
(3) Find the projection of $\left[H, \sum_i g_i S_i\right]$ onto a component μ_{ijk} by taking the trace of the basis element μ_{ijk} with the result of (2). In other words, calculate $\mathrm{tr}\left(\left[H, \sum_i g_i S_i\right]\mu_{ijk}\right)$ for each μ_{ijk}.
(4) Set all of the projections equal to zero and then solve the system of linear equations for the expansion coefficients a_{ijk} which satisfy these relations, thereby determining the H which will commute with $\sum_i g_i S_i$.

The results are summarized in the following:

$$h_1 = \sum_{i=1}^{8} \mu_{0ii}, \tag{26}$$

$$h_2 = \sum_{i=1}^{8} \mu_{i0i}, \tag{27}$$

$$h_3 = \sum_{i=1}^{8} \mu_{ii0}, \tag{28}$$

$$h_4 = \sum_{ijk \neq 0} f_{ijk}\mu_{ijk}, \tag{29}$$

$$h_5 = \sum_{ijk \neq 0} d_{ijk}\mu_{ijk}, \tag{30}$$

where the f_{ijk} are the structure constants and the d_{ijk} are components of the totally symmetric d-tensor (see appendix A). These are the only Hamiltonians, along with combinations of these, which commute with all of the collective errors that may act on a set of three qutrits. The first three are generalizations of the Heisenberg exchange operations which are known to be universal for the three- and four-qubit DFS/NSs [8]. The last two are, perhaps, not immediately obvious candidates for DFS compatible Hamiltonians. However, we will show analytically that these two, along with the first three, are compatible with all three *qudit* NSs.

4.2. Hamiltonians compatible with a three-qudit DFS

We will now show that a set of Hamiltonians having the form of equations (26)–(30) are compatible with a three-*qudit* DFS/NS and can be obtained analytically. This follows from the fact that equation (A.2), the Jacobi identity equation (A.3) and Jacobi-like identity equation (A.4) hold for all $SU(d)$, $d \geqslant 3$ [5].

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[5] Actually the equation also holds for $SU(2)$ with $d_{ijk} = 0$. However, the nonzero d_{ijk} make all the difference.

Let us consider a Hamiltonian constructed from the μ_{ijk} with the $N = d^2 - 1$ constituent basis elements λ_i belonging to a matrix representation of the Lie algebra of $SU(d)$ with d arbitrary:

$$H = \sum_{ijk} a_{ijk}\mu_{ijk}. \tag{31}$$

The three qutrit case above is a special case for $SU(d)$ when $d = 3$. If we consider collective errors on the DFS/NS,

$$S_j = \mu_{j00} + \mu_{0j0} + \mu_{00j},$$

our condition, equation (25) reads

$$[S_j, H] = 0,$$

which implies that we require

$$f_{ilm}a_{jkl} + f_{jlm}a_{kil} + f_{klm}a_{ijl} = 0. \tag{32}$$

Comparing this equation with equations (A.3) and (A.4), we can immediately see that there are two sets of numbers a_{ijk} which satisfy equation (32), f_{ijk} and d_{ijk}. In other words, this equation is satisfied for $a_{ijk} = f_{ijk}$ and for $a_{ijk} = d_{ijk}$.

To obtain Hamiltonians of the form h_1, h_2, h_3, we let one of the factors be the identity in μ_{ijk}. Let us consider one particular case, the case when the third factor is the identity

$$H = \sum_{i,j=1}^{N} a_{ij0}\mu_{ij0}$$

and calculate $[H, S_j]$. The result is

$$a_{il0}f_{lkj} + a_{lj0}f_{lki} = 0.$$

If we multiply by d_{mki} and sum over k and i, we get

$$a_{il0}f_{lkj}d_{mki} = 0.$$

Clearly, if $a_{il0} = \delta_{il}$ this equation is satisfied and we obtain the form equation (28). Similarly for equations (26) and (27). We emphasize that the analytic proof is valid for all three-qudit DFS/NSs, not just qutrit states.

Also, we have not proved the converse; i.e., we have not shown that these are the only Hamiltonians which commute with collective errors on three qudits. The algorithm in the previous section has been implemented for qutrits and, in that case, these are the only Hamiltonians satisfying the stated conditions for DFS/NS compatibility. We suspect that this is also the case for qudits.

4.3. Generalizations

It is important to note that several of these results are quite general. First, we reiterate that the algorithm used to find the Hamiltonians for the collective three qutrit DFS/NS can be used for any DFS/NS given a basis for the stabilizer elements of the code space. Second, as shown later, due to the permutation symmetry of qudits undergoing collective errors, the Hamiltonians we have derived are compatible with any qudit DFS/NS undergoing collective errors. Perhaps even more generally stated, the d-state system Hamiltonians with the same form as equations (26)–(30), including the generalized exchange interaction, commute with any collective transformation on a set of qudits.

Furthermore, as shown in the following section, the exchange Hamiltonians can be analytically exponentiated to provide generalized SWAP operations. The SWAP generates the permutation group on the set of qudits and this group commutes with the group of collective unitary transformations. A description of this is given in [30] where it is shown that one can use this relation to produce efficient qudit circuits.

5. Universal computation

We may now ask the question, are these Hamiltonians sufficient to perform arbitrary unitary transformations on NS qubits? In other words, can we perform universal computation on this NS using the compatible Hamiltonians provided here?

In order to answer these questions, we will first note some important properties of the algebra of the Hamiltonians we have found to be DFS-compatible. Then we attempt to find the corresponding unitary transformations.

5.1. Commutation relations

Now let $N = d^2 - 1$ be the number of matrices in a basis for the Lie algebra of $SU(d)$. We will define the d-dimensional analogs of equations (26)–(30) by

$$e_1(d) = \sum_{i=1}^{N} \mu_{0ii}, \tag{33}$$

$$e_2(d) = \sum_{i=1}^{N} \mu_{i0i}, \tag{34}$$

$$e_3(d) = \sum_{i=1}^{N} \mu_{ii0}, \tag{35}$$

$$F(d) = \sum_{ijk \neq 0} f_{ijk} \mu_{ijk}, \tag{36}$$

$$D(d) = \sum_{ijk \neq 0} d_{ijk} \mu_{ijk}. \tag{37}$$

where the μ_{ijk} now represent the tensor product of three $d \times d$ basis matrices. It can be shown that the commutation of any two different $e_i(d)$ Hamiltonians yields the $F(d)$ Hamiltonian,

$$[e_1(d), e_2(d)] = -2\,\mathrm{i} F(d), \tag{38}$$

$$[e_1(d), e_3(d)] = +2\,\mathrm{i} F(d), \tag{39}$$

$$[e_2(d), e_3(d)] = -2\,\mathrm{i} F(d). \tag{40}$$

One can also show that the commutation of an $e_i(d)$ Hamiltonian with the $F(d)$ Hamiltonian gives a combination of the remaining two e_i's:

$$[e_1(d), F(d)] = 4\,\mathrm{i}(e_2(d) - e_3(d)), \tag{41}$$

$$[e_2(d), F(d)] = 4\,\mathrm{i}(e_3(d) - e_1(d)), \tag{42}$$

$$[e_3(d), F(d)] = 4\,\mathrm{i}(e_1(d) - e_2(d)). \tag{43}$$

Furthermore, we have found that the $D(d)$ Hamiltonian commutes with the three e_i's as well as $F(d)$,

$$[e_i(d), D(d)] = 0, \qquad i = 1, 2, 3 \tag{44}$$

and

$$[F(d), D(d)] = 0. \tag{45}$$

This last relation, equation (45), can be obtained by an explicit expansion of the two ordered products using equation (A.2). The expansion may be reduced to terms involving products of the form $f_{ijm} f_{klm}$ which can be expanded using equation (A.9). What remains can be reduced further with the help of equations (A.12) and (A.13) giving the stated result.

With these results, we may now show that a sub-algebra isomorphic to the Lie algebra of $SU(2)$ is generated by a combination of these Hamiltonians. First, note that from equations (41) and (42), we may show that

$$[(e_1 - e_2), F] = 4\,\mathrm{i}(e_1 + e_2 - 2e_3). \tag{46}$$

From equations (38)–(40)

$$[(e_1 - e_2), (e_1 + e_2 - 2e_3)] = -12\,\mathrm{i}F. \tag{47}$$

Therefore, the three matrices $(e_1 - e_2)/2\sqrt{3}$, $(e_1 + e_2 - 2e_3)/6$ and $F/2\sqrt{3}$ form a representation of the Lie algebra of $SU(2)$. We will now use this result in the construction of the logical analogues of the Pauli matrices acting on the encoded qubit states.

5.2. Unitary transformations and logical operations for qudits

Before obtaining the logical gating operations, it is interesting to note that analytic expressions for the exponential of each one of the three e_j Hamiltonians in the physical basis may be obtained for qudits. There are two fortuitous properties of these matrices which enable us to provide such an analytic expression: (1) the sum of the off-diagonal matrices commutes with the sum of the diagonal matrices and (2) when squared, the sum of the off-diagonal matrices is diagonal and can easily be summed. This can be shown to be true by direct computation using the set of Gell–Mann matrices in the case of qutrits. Appendix B provides proof that it is also true for qudits. These two properties enable us to sum the series resulting from the exponential of the Hamiltonians e_1, e_2, e_3. An explicit form for the unitary evolution corresponding to these Hamiltonians is also given in appendix B.

Now let us define the logical 'X' operator, **X**, which acts on the DFS through the relation

$$\bar{\mathbf{X}} = \frac{1}{2\sqrt{3}}(e_1 - e_2). \tag{48}$$

We note that the overall sign of the states spanning logical one (equation (19)) are chosen so that the form of $\bar{\mathbf{X}}$ above resembles the expressions appearing in equations (14) and (10) for the three- and four-qubit DFSs, respectively. It should also be mentioned that we are describing the logical X operation in terms of Hamiltonians rather than unitary transformations. For comparison, note that equations (14) and (10) could also be written as $\frac{1}{2\sqrt{3}}(\vec{\sigma}_2 \cdot \vec{\sigma}_3 - \vec{\sigma}_1 \cdot \vec{\sigma}_3)$.

The exponentiation of $\bar{\mathbf{X}}$ leads to a time evolution given by

$$U_{\bar{\mathbf{X}}} = \mathbb{1} + \mathrm{i}\bar{\mathbf{X}}\sin(t) - \bar{\mathbf{X}}^2(1 - \cos(t)). \tag{49}$$

This can be obtained in two different ways. One way is to use the prescription for the exponential of the e_i as described above. Another one is to calculate the exponential from the

results of appendix B.3 and then transform back to the physical basis using the fact that for a similarity transformation V, and a Hamiltonian H,

$$V \exp(-iHt)V^{-1} = \exp(-iVHV^{-1}t). \tag{50}$$

This enables one to transform between the physical, or computational basis states and the logical basis states as well as between the different sets of operators—the physical unitary transformations one would implement in experiments and the logical ones.

Similarly, the logical 'Z' operator, $\mathbf{Z}$, can be expanded in terms of the e_i's by

$$\bar{\mathbf{Z}} = \tfrac{1}{6}(e_1 + e_2 - 2e_3). \tag{51}$$

The form of the unitary transformation resulting from the exponentiation of $\bar{\mathbf{Z}}$ is similar to that of $U_{\bar{\mathbf{X}}}$ and is given by

$$U_{\bar{\mathbf{Z}}} = \mathbb{1} - i\bar{\mathbf{Z}}\sin(t) - \bar{\mathbf{Z}}^2(1 - \cos(t)). \tag{52}$$

Once $\bar{\mathbf{X}}$ and $\bar{\mathbf{Z}}$ are found the logical $\bar{\mathbf{Y}}$ can be produced through commutation. This shows, with our particular choice of scaling, that the three matrices proportional to $(e_1 + e_2 - 2e_3)$, $(e_2 - e_1)$ and F form a representation of the Lie algebra of $SU(2)$ and act as the Pauli matrices on the DFS qubit made from three qutrits.

A rotation in $SU(2)$ about an arbitrary axis can be obtained by three successive rotations. In particular, a rotation about the logical $\bar{\mathbf{Y}}$ may be performed using the following decomposition of the logical $SU(2)$ group, using real (Euler) angles α, β and γ:

$$U(\alpha, \beta, \gamma) = \exp[-i\bar{\mathbf{Z}}\alpha]\exp[-i\bar{\mathbf{X}}\beta]\exp[-i\bar{\mathbf{Z}}\gamma]. \tag{53}$$

Indeed this parametrizes all of the group $SU(2)$. Therefore, we have shown that for an encoded qubit comprised of three qudits, these DFS/NS compatible operations alone can perform any rotation over the logical subsystems.

5.3. Unitary transformations and logical operations for qutrits

In the logical basis, $U_{\bar{\mathbf{X}}}$ acts as the identity on the singlet and decuplet states (given in equation (16), and explicitly in [58]) throughout the entire evolution. The action of $\bar{\mathbf{X}}$ on the states forming the logical zero (equations (18) is such that it swaps the states for their logical one counterparts equations (19)) and vice versa,

$$\bar{\mathbf{X}}\psi_j^{8,0} = \psi_j^{8,1}, \tag{54}$$

$$\bar{\mathbf{X}}\psi_j^{8,1} = \psi_j^{8,0}. \tag{55}$$

In other words, it acts as a Pauli X gate on the logical states. These relations can be used along with equation (49) to calculate the action of $U_{\bar{\mathbf{X}}}$ on logical zero basis states,

$$\begin{aligned}
U_{\bar{\mathbf{X}}}\big(\psi_j^{8,0}\big) &= [\mathbb{1} + i\bar{\mathbf{X}}\sin(t) - \bar{\mathbf{X}}^2(1 - \cos(t))]\psi_j^{8,0} \\
&= \psi_j^{8,0} + i\sin(t)\psi_j^{8,1} - (1 - \cos(t))\psi_j^{8,0} \\
&= \cos(t)\psi_j^{8,0} + i\sin(t)\psi_j^{8,1}.
\end{aligned} \tag{56}$$

It therefore follows that

$$U_{\bar{\mathbf{X}}}|0_L\rangle = \cos(t)|0_L\rangle + i\sin(t)|1_L\rangle, \tag{57}$$

and similarly,

$$U_{\bar{\mathbf{X}}}|1_L\rangle = \cos(t)|1_L\rangle + i\sin(t)|0_L\rangle. \tag{58}$$

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Now note that the logical 'Z' operator, $\bar{\mathbf{Z}}$, acts as the identity on states in octet 1 while changing the overall sign of states in octet 2 and this can be used to obtain equation (51) directly for qutrits using the explicit expressions. The unitary may then also be obtained directly from $\bar{\mathbf{Z}}$. The transformation of logical zero is given by

$$
\begin{aligned}
U_{\bar{\mathbf{Z}}}|0_L\rangle &= U_{\bar{\mathbf{Z}}} \sum_j \alpha_j \psi_j^{8,0} = [\mathbb{1} - \mathrm{i}\bar{\mathbf{Z}}\sin(t) - \bar{\mathbf{Z}}^2(1-\cos(t))] \sum_j \alpha_j \psi_j^{8,0} \\
&= \sum_j \alpha_j \psi_j^{8,0} - \mathrm{i}\sin(t) \sum_j \alpha_j \psi_j^{8,0} - (1-\cos(t)) \sum_j \alpha_j \psi_j^{8,0} \\
&= |0_L\rangle \exp(-\mathrm{i}t),
\end{aligned}
\tag{59}
$$

while logical one transforms unitarily by

$$
U_{\bar{\mathbf{Z}}}|1_L\rangle = |1_L\rangle \exp(+\mathrm{i}t).
\tag{60}
$$

Again, the decuplet states are left unchanged by the action of $U_{\bar{\mathbf{Z}}}$. This implies, along with the invariance of the singlet state, that this gate set is canonical in the sense described in [62] which is important for applications of decoupling pulses to eliminate leakage and protect the information [61, 62].

5.4. SWAP operation

The exchange, or SWAP operation between qudits p and q can be achieved by allowing the appropriate DFS/NS compatible Hamiltonian $e_m(d)$, ($p \neq m \neq q$) to act between the two for a specific amount of time. To show this, let us write

$$
\sum_{s=1}^{d^2-1} \lambda_s \otimes \lambda_s = \sum_i \lambda_i \otimes \lambda_i + \sum_j \lambda_j \otimes \lambda_j,
\tag{61}
$$

where the λ_i (λ_j) represent the diagonal (off-diagonal) components of the traceless, Hermitian basis $\{\lambda_s\}$ normalized such that $\mathrm{Tr}(\lambda_s \lambda_{s'}) = 2\delta_{ss'}$. (*In this section only, we use i for indices on diagonal elements of the algebra and j for elements of the algebra which have no nonzero diagonal elements.*) Since $\sum_i \lambda_i \otimes \lambda_i$ commutes with $\sum_j \lambda_j \otimes \lambda_j$ (see appendix B.1) we may express the exponential of $\sum_{s=1}^{d^2-1} \lambda_s \otimes \lambda_s$ as

$$
\exp\left[-\mathrm{i}t \sum_{s=1}^{d^2-1} \lambda_s \otimes \lambda_s\right] = \exp\left[-\mathrm{i}t \sum_i \lambda_i \otimes \lambda_i\right] \exp\left[-\mathrm{i}t \sum_j \lambda_j \otimes \lambda_j\right].
\tag{62}
$$

The off-diagonal members of this basis may be written either as $|k\rangle\langle l| + |l\rangle\langle k|$ or $\mathrm{i}|k\rangle\langle l| - \mathrm{i}|l\rangle\langle k|$ for $k \neq l$ (for notational simplicity we use the two-qudit computational basis ($\{|11\rangle, |12\rangle, \ldots, |dd\rangle\}$)), so that the off-diagonal contribution appearing in this last equation may be written as

$$
\exp\left[-\mathrm{i}t \sum_j \lambda_j \otimes \lambda_j\right] = \exp\left[-\mathrm{i}t \sum_{k<l} M_{k,l}\right],
\tag{63}
$$

where

$$
\begin{aligned}
M_{k,l} &\equiv (|k\rangle\langle l| + |l\rangle\langle k|) \otimes (|k\rangle\langle l| + |l\rangle\langle k|) + (\mathrm{i}|k\rangle\langle l| - \mathrm{i}|l\rangle\langle k|) \otimes (\mathrm{i}|k\rangle\langle l| - \mathrm{i}|l\rangle\langle k|) \\
&= 2|k\rangle\langle l| \otimes |l\rangle\langle k| + 2|l\rangle\langle k| \otimes |k\rangle\langle l|.
\end{aligned}
\tag{64}
$$

For $SU(d)$ the number of $M_{k,l}$ appearing in the summation of equation (63) is $(d^2-d)/2$. Also, each distinct $M_{k,l}$ can be seen to commute with the others since $(|k\rangle\langle l| \otimes |l\rangle\langle k| + |l\rangle\langle k| \otimes$

$|k\rangle\langle l|) \times (|p\rangle\langle q| \otimes |q\rangle\langle p| + |q\rangle\langle p| \otimes |p\rangle\langle q|) = 0$ when $p \neq l \neq q$ and $p \neq k \neq q$. This allows us to write

$$\exp\left[-\mathrm{i}t \sum_j \lambda_j \otimes \lambda_j\right] = \exp[-\mathrm{i}t\,M_{1,2}] \cdots \exp[-\mathrm{i}t\,M_{d-1,d}]. \tag{65}$$

When squared, each $M_{k,l}$ is diagonal and given by $M_{k,l}^2 = 4|k\rangle\langle k| \otimes |l\rangle\langle l| + 4|l\rangle\langle l| \otimes |k\rangle\langle k|$, which implies that $M_{k,l}^3 = 4M_{k,l}$, etc. Now, if we choose to define

$$Q_{k,l} \equiv \frac{M_{k,l}}{2} = |k\rangle\langle l| \otimes |l\rangle\langle k| + |l\rangle\langle k| \otimes |k\rangle\langle l|, \qquad l \neq k \tag{66}$$

and

$$R_{k,l} \equiv \frac{M_{k,l}^2}{4} = |k\rangle\langle k| \otimes |l\rangle\langle l| + |l\rangle\langle l| \otimes |k\rangle\langle k|, \qquad l \neq k \tag{67}$$

we may express the exponential of $M_{k,l}$ as

$$U_{k,l}(t) = \exp[-\mathrm{i}t\,M_{k,l}] = \mathbb{1} - \mathrm{i}Q_{k,l}\sin(2t) + R_{k,l}(\cos(2t) - 1). \tag{68}$$

If we now let $t = \pi/4$ we find that

$$U_{k,l}(t = \pi/4) = \mathbb{1} - \mathrm{i}Q_{k,l} - R_{k,l}. \tag{69}$$

Using the definition provided in equation (66) we see that the only two-qudit computational basis states $|\alpha\beta\rangle$ which survive the action of a particular $Q_{k,l}$ are $|lk\rangle$ and $|kl\rangle$, i.e.,

$$Q_{k,l}|\alpha\beta\rangle = \delta_{\alpha,l}\delta_{\beta,k}|kl\rangle + \delta_{\alpha,k}\delta_{\beta,l}|lk\rangle. \tag{70}$$

Also, using equation (67), we find that

$$(\mathbb{1} - R_{k,l})|\alpha\beta\rangle = |\alpha\beta\rangle - \delta_{\alpha,l}\delta_{\beta,k}|\alpha\beta\rangle - \delta_{\alpha,k}\delta_{\beta,l}|\alpha\beta\rangle. \tag{71}$$

Therefore, at $t = \pi/4$ the two-qudit computational basis states $|\alpha\beta\rangle$ evolve to

$$U_{k,l}(\pi/4)|\alpha\beta\rangle = |\alpha\beta\rangle - \delta_{\alpha,l}\delta_{\beta,k}(\mathrm{i}|kl\rangle + |\alpha\beta\rangle) - \delta_{\alpha,k}\delta_{\beta,l}(\mathrm{i}|lk\rangle + |\alpha\beta\rangle). \tag{72}$$

In other words, $U_{k,l}(\pi/4)$ acts as the identity on all product states $|\alpha\beta\rangle$ except $|kl\rangle$ and $|lk\rangle$. On these states $U_{k,l}(\pi/4)$ exchanges $|kl\rangle$ ($|lk\rangle$) for $|lk\rangle$ ($|kl\rangle$) along with a phase shift of $\exp(-\mathrm{i}\pi/2)$. The action of $\exp\left[-\mathrm{i}t \sum_{s=1}^{d^2-1} \lambda_s \otimes \lambda_s\right]$ at $t = \pi/4$ on a given product state $|\alpha\beta\rangle$, $\alpha, \beta = 1, 2, \ldots, d$ can now be calculated using equations (62) and (65)

$$\exp\left[-\mathrm{i}(\pi/4)\sum_{s=1}^{d^2-1}\lambda_s \otimes \lambda_s\right]|\alpha\beta\rangle = \begin{cases} \exp\left(-\mathrm{i}(\pi/4)\sum_i \lambda_i \otimes \lambda_i\right)|\alpha\alpha\rangle, & \text{if } \alpha = \beta, \\ -\mathrm{i}\exp\left(-\mathrm{i}(\pi/4)\sum_i \lambda_i \otimes \lambda_i\right)|\beta\alpha\rangle, & \text{if } \alpha \neq \beta. \end{cases} \tag{73}$$

In order to determine the diagonal elements of $\exp\left(-\mathrm{i}t \sum_i \lambda_i \otimes \lambda_i\right)$ we must first find the coefficients $\xi_{m,n}$ which satisfy the relation

$$\sum_i \lambda_i \otimes \lambda_i = \sum_{m,n=1}^d \xi_{m,n}|m\rangle\langle m| \otimes |n\rangle\langle n|. \tag{74}$$

In the following analysis we will take a Gell–Mann basis for $SU(d)$, in which there are a total of $d-1$ diagonal elements. All of these diagonal matrices can be constructed using equation (B.8). For $SU(3)$ the two diagonal elements are given by $\lambda_3 = |1\rangle\langle 1| - |2\rangle\langle 2|$ and $\lambda_8 = (|1\rangle\langle 1| + |2\rangle\langle 2| - 2|3\rangle\langle 3|)/\sqrt{3}$. Two of the three diagonal elements of $SU(4)$ can be formed by respectively placing λ_3 and λ_8 in the upper left block of a 4×4 matrix with zero entries on the fourth row and column. The remaining diagonal element is then given by equation (B.8) with $d = 4$, i.e., $\lambda_{15} = (|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| - 3|4\rangle\langle 4|)/\sqrt{6}$. This procedure

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can be used to construct the set of diagonal matrices corresponding to $SU(d)$ for all $d \geqslant 3$; one simply places the diagonal elements associated with $SU(d-1)$ into the upper left block of a $d \times d$ matrix with zeros along the dth row and column. Equation (B.8) can be used to obtain the remaining element.

As we proceed to determine the expansion coefficients $\xi_{m,n}$ appearing in equation (74) let us now write the sum of off-diagonal matrices explicitly as

$$\sum_i \lambda_i \otimes \lambda_i = (|1\rangle\langle 1| - |2\rangle\langle 2|) \otimes (|1\rangle\langle 1| - |2\rangle\langle 2|) + \cdots$$

$$+ \frac{2}{d(d-1)} (|1\rangle\langle 1| + \cdots + |d-1\rangle\langle d-1| - (d-1)|d\rangle\langle d|)$$
$$\otimes (|1\rangle\langle 1| + \cdots + |d-1\rangle\langle d-1| - (d-1)|d\rangle\langle d|). \tag{75}$$

If we consider the case when $m = n$ we find that the coefficient $\xi_{m,m}$ attached to the matrix element $|m\rangle\langle m| \otimes |m\rangle\langle m|$ obeys the relation

$$\xi_{m,m} = \begin{cases} \dfrac{2(d-1)}{d}, & \text{if} \quad m = d, \\ \dfrac{2(m-1)}{m} + 2 \displaystyle\sum_{\mu=m+1}^{d} \dfrac{1}{\mu(\mu-1)}, & \text{if} \quad m \neq d. \end{cases} \tag{76}$$

Let us now evaluate the series which appears in the expression for $\xi_{m,m}$ when $m \neq d$.

$$\sum_{\mu=m+1}^{d} \frac{1}{\mu(\mu-1)} = \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} + \cdots + \frac{1}{(d-2)(d-1)} + \frac{1}{(d-1)d}$$

$$= \frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+1} - \cdots - \frac{1}{d-1} + \frac{1}{d-1} - \frac{1}{d}$$

$$= \frac{1}{m} - \frac{1}{d} = \frac{d-m}{md}. \tag{77}$$

Therefore, when $m \neq d$ we find that the coefficient $\xi_{m,m}$ can be expressed as

$$\frac{2(m-1)}{m} + 2 \sum_{\mu=m+1}^{d} \frac{1}{\mu(\mu-1)} = \frac{2(m-1)}{m} + \frac{2(d-m)}{md} = \frac{2(d-1)}{d}, \tag{78}$$

so that

$$\xi_{m,m} = \frac{2(d-1)}{d}, \qquad \text{for} \quad m = 1, 2, \ldots, d. \tag{79}$$

Now suppose that $m \neq n$. Examination of the expansion given by equation (75) reveals the symmetry of all terms $|m\rangle\langle m| \otimes |n\rangle\langle n|$ under the interchange of indices m and n. Since the relation $\xi_{m,n} = \xi_{n,m}$ then follows, we may assume that $n > m$ in the proceeding discussion without loss of generality. Given this assumption, we find that the coefficient $\xi_{m,n}$ appearing in equation (74) satisfies the equation

$$\xi_{m,n} = \begin{cases} -\dfrac{2}{d}, & \text{if} \quad n = d, \\ -\dfrac{2}{n} + 2 \displaystyle\sum_{\mu=n+1}^{d} \dfrac{1}{\mu(\mu-1)}, & \text{if} \quad n \neq d. \end{cases} \tag{80}$$

However, this relation for $\xi_{m,n}$ when $n \neq d$ may be simplified in view of equation (77),

$$-\frac{2}{n} + 2 \sum_{\mu=n+1}^{d} \frac{1}{\mu(\mu-1)} = -\frac{2}{n} + \frac{2(d-n)}{nd} = -\frac{2}{d}. \tag{81}$$

And so we find that all of the coefficients $\xi_{m,n}$ ($m \neq n$) are equal and given by

$$\xi_{m,n} = -\frac{2}{d}, \qquad \text{for} \quad m, n = 1, 2, \ldots, d \quad \text{and} \quad m \neq n. \tag{82}$$

The exponential of $\sum_i \lambda_i \otimes \lambda_i = \sum_{m,n=1}^d \xi_{m,n} |m\rangle\langle m| \otimes |n\rangle\langle n|$ can now be evaluated,

$$
\begin{aligned}
\exp\left[-\mathrm{i}t \sum_i \lambda_i \otimes \lambda_i\right] &= \sum_{m,n=1}^d \exp(-\mathrm{i}t\xi_{m,n}) |m\rangle\langle m| \otimes |n\rangle\langle n| \\
&= \sum_m \exp(-(2\mathrm{i}t(d-1))/d) |m\rangle\langle m| \otimes |m\rangle\langle m| \\
&\quad + \sum_{m\neq n} \exp(2\mathrm{i}t/d) |m\rangle\langle m| \otimes |n\rangle\langle n|.
\end{aligned} \tag{83}
$$

Therefore, the unitary transformation which corresponds to the exponentiation of $\sum_{s=1}^{d^2-1} \lambda_s \otimes \lambda_s$ can be expressed using equations (62), (65), (68) and (83). In particular, the two-qudit computational basis states $|\alpha\beta\rangle$, $\alpha, \beta = 1, 2, \ldots, d$ evolve at $t = \pi/4$ to

$$\exp\left[-\mathrm{i}(\pi/4) \sum_{s=1}^{d^2-1} \lambda_s \otimes \lambda_s\right] |\alpha\beta\rangle = \begin{cases} \exp(-(\pi\mathrm{i}(d-1))/2d)|\alpha\alpha\rangle, & \text{if} \quad \alpha = \beta, \\ -\mathrm{i}\exp(\pi\mathrm{i}/2d)|\beta\alpha\rangle, & \text{if} \quad \alpha \neq \beta. \end{cases} \tag{84}$$

However, since $\exp(-(\pi\mathrm{i}(d-1))/2d) = \exp(-\pi\mathrm{i}/2)\exp(\pi\mathrm{i}/2d) = -\mathrm{i}\exp(\pi\mathrm{i}/2d)$, we have

$$\exp\left[-\mathrm{i}(\pi/4) \sum_{s=1}^{d^2-1} \lambda_s \otimes \lambda_s\right] |\alpha\beta\rangle = -\mathrm{i}\exp(\pi\mathrm{i}/2d)|\beta\alpha\rangle \quad \text{for} \quad \alpha, \beta = 1, 2, \ldots, d. \tag{85}$$

Although the exponential of $\sum_{s=1}^{d^2-1} \lambda_s \otimes \lambda_s$ induces a phase shift on all product states $|\alpha\beta\rangle$, the shift is the same for all initial states as they evolve to $t = \pi/4$. This unitary transformation thus acts as the SWAP operation at $t = \pi/4$ between any two d-state systems.

6. Hamiltonians compatible with an *n*-qudit DFS

In this section we show that the Hamiltonians given by equations (33)–(37) remain compatible with a collective DFS/NS encoding of n physical qudits into a set of logical quDits, $d \neq D$. Since, in principle, a logical *quDit* of dimension D can be formed by such an encoding, the Hamiltonians considered here have the ability to generate compatible transformations on an encoded qudit state of arbitrary dimension. To see this, let us first consider the exchange operations $e_i(d)$ which were previously shown to commute with the collective errors, i.e.,

$$
\begin{aligned}
[e_1(d), S_j] &= [\mathbb{1} \otimes \lambda_i \otimes \lambda_i \, , \, \lambda_j \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \lambda_j \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \lambda_j] \\
&= \mathbb{1} \otimes (\lambda_i\lambda_j - \lambda_j\lambda_i) \otimes \lambda_i + \mathbb{1} \otimes \lambda_i \otimes (\lambda_i\lambda_j - \lambda_j\lambda_i) \\
&= 2\mathrm{i} f_{ijk}(\mathbb{1} \otimes \lambda_k \otimes \lambda_i - \mathbb{1} \otimes \lambda_k \otimes \lambda_i) \\
&= 0.
\end{aligned} \tag{86}
$$

(Recall that the structure constants f_{ijk} are totally antisymmetric and that the generators of $SU(d)$ obey the commutation relations $[\lambda_i, \lambda_j] = 2\mathrm{i} f_{ijk}\lambda_k$. Again, summation over repeated indices is implied.) Note how the term $[\lambda_i \otimes \lambda_i \otimes \mathbb{1}, \mathbb{1} \otimes \mathbb{1} \otimes \lambda_j]$ in the commutator of $[e_1(d), S_j]$ can be immediately neglected while the nontrivial contributions are proportional

to $[\lambda_i, \lambda_j] \otimes \lambda_i + \lambda_i \otimes [\lambda_i, \lambda_j]$. The commutators of $e_2(d)$ and $e_3(d)$ with the collective errors also share these properties,

$$
\begin{aligned}
[e_2(d), S_j] &= [\lambda_i \otimes \mathbb{1} \otimes \lambda_i, \ \lambda_j \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \lambda_j \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \lambda_j] \\
&= [\lambda_i, \lambda_j] \otimes \mathbb{1} \otimes \lambda_i + \lambda_i \otimes \mathbb{1} \otimes [\lambda_i, \lambda_j] \\
&= 2\mathrm{i} f_{ijk}(\lambda_k \otimes \mathbb{1} \otimes \lambda_i - \lambda_k \otimes \mathbb{1} \otimes \lambda_i) \\
&= 0,
\end{aligned}
\tag{87}
$$

and similarly for $[e_3(d), S_j]$. We now allow the system to be composed of n physical qudits. Define

$$
e_{pq} = \sum_i \lambda_i^{(p)} \lambda_i^{(q)},
\tag{88}
$$

where the superscripts identify qudits p and q, all other qudits are acted upon by identity operators, and we have written the sum explicitly for emphasis. We can now show that all of the generalized exchange Hamiltonians e_{pq} for this larger system also commute with the collective errors S_α. To see this, note that the collective errors acting on the n qudits are again defined as

$$
S_\alpha \equiv \sum_{r=1}^n \lambda_\alpha^{(r)}, \qquad \text{for} \quad \alpha = 1, 2, \ldots, d^2 - 1.
\tag{89}
$$

The only nontrivial contributions appearing in the calculation of $[e_{pq}, S_\alpha]$ have the form

$$
[\lambda_i, \lambda_j]^{(p)} \lambda_i^{(q)} + \lambda_i^{(p)} [\lambda_i, \lambda_j]^{(q)}
\tag{90}
$$

with the identity lying at all other positions. Since these terms may be rewritten as

$$
2\mathrm{i} f_{ijk}\big(\lambda_k^{(p)} \lambda_i^{(q)} + \lambda_i^{(p)} \lambda_k^{(q)}\big),
\tag{91}
$$

or

$$
2\mathrm{i} f_{ijk}\big(\lambda_k^{(p)} \lambda_i^{(q)} - \lambda_k^{(p)} \lambda_i^{(q)}\big) = 0,
\tag{92}
$$

we see that the exchange Hamiltonians are also compatible with a DFS/NS that is supported by n physical qudits.

This generalization to n qudits is also applicable to the Hamiltonian $F(d)$. Recall the commutator of $F(d)$ with a three particle collective error. In this case we have

$$
\begin{aligned}
[F(d), S_l] &= [f_{ijk}\lambda_i \otimes \lambda_j \otimes \lambda_k, \ \lambda_l \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \lambda_l \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \lambda_l] \\
&= f_{ijk}([\lambda_i, \lambda_l] \otimes \lambda_j \otimes \lambda_k + \lambda_i \otimes [\lambda_j, \lambda_l] \otimes \lambda_k + \lambda_i \otimes \lambda_j \otimes [\lambda_k, \lambda_l]) \\
&= 2\mathrm{i} f_{ijk}(f_{ilm}\lambda_m \otimes \lambda_j \otimes \lambda_k + f_{jlm}\lambda_i \otimes \lambda_m \otimes \lambda_k + f_{klm}\lambda_i \otimes \lambda_j \otimes \lambda_m).
\end{aligned}
\tag{93}
$$

This can be reduced using equation (A.9). After some relabeling we obtain

$$
\begin{aligned}
[F(d), S_l] &= 2\mathrm{i}(d_{jlk}d_{kmi} - d_{jlk}d_{kmi} + d_{jik}d_{kml} \\
&\quad - d_{jik}d_{kml} + d_{jmk}d_{kli} - d_{jmk}d_{kli})\,\lambda_i \otimes \lambda_j \otimes \lambda_m \\
&= 0.
\end{aligned}
\tag{94}
$$

Now suppose that we have a system of n qudits with the interaction coupling particles p, q and r, then

$$
[F(d), S_l] = \big[f_{ijk}\lambda_i^{(p)}\lambda_j^{(q)}\lambda_k^{(r)}, \ \lambda_l \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \lambda_l\big].
\tag{95}
$$

Again, all of the terms in this expansion that do not contain ordinary products of two λ's will cancel trivially, those which remain can be expressed as

$$
f_{ijk}\big([\lambda_i, \lambda_l]^{(p)}\lambda_j^{(q)}\lambda_k^{(r)} + \lambda_i^{(p)}[\lambda_j, \lambda_l]^{(q)}\lambda_k^{(r)} + \lambda_i^{(p)}\lambda_j^{(q)}[\lambda_k, \lambda_l]^{(r)}\big),
\tag{96}
$$

which becomes

$$
\begin{aligned}
[F(d), S_l] &= 2\mathrm{i}(d_{jlk}d_{kmi} - d_{jlk}d_{kmi} + d_{jik}d_{kml} \\
&\quad - d_{jik}d_{kml} + d_{jmk}d_{kli} - d_{jmk}d_{kli})\, \lambda_i^{(p)}\lambda_j^{(q)}\lambda_m^{(r)} \\
&= 0.
\end{aligned}
\tag{97}
$$

The $D(d)$ Hamiltonian can also be shown in a similar way to commute with the collective errors acting on a system of n qudits. Therefore, these DFS/NS compatible operators can be used to manipulate encoded qudits of arbitrary dimension as well.

7. Conclusions

We have presented a set of Hamiltonians that are compatible with an n-qudit DFS/NS (d and n arbitrary) encoding which protects against collective noise, i.e., any noise which affects the qudits in the same way. We have also provided a set of unitary transformations which can be used to perform any rotation on a logical qubit that is represented by the two degenerate representations in the product space of three qudits. The generalized exchange operations are analogues of the Heisenberg exchange transformation which is known to be universal for various systems of qubits and we have shown that they, as well as the logical gating operations, can be exponentiated analytically. In this context, we note that the three-qudit DFS/NS is similar in structure to the three qubit DFS/NS.

Our analysis has focused on Hamiltonians which commute with elements of the stabilizer and the unitary transformations arising from the exponentiation of the generalized exchange operation for qudits. For three qutrits we have found the complete set of Hamiltonians satisfying this commutation condition. However, these results could be extended by using the less restrictive requirement provided in equation (22).

In order for these Hamiltonians to enable universal quantum computation they must also be able to generate entanglement between two encoded qubits. The CNOT gate acting on a subspace of the three-qubit DF-subsystem was provided in [38] using a circuit of 19 exchange interactions. Since the states considered there also appear in the three-qutrit DFS/NS as $\psi_2^{8,0}$ and $\psi_2^{8,1}$ we find that the exchange interaction alone can implement universal quantum computing over the three-qutrit DFS/NS as well. Furthermore, since the tableau for all three-*qudit* DFSs have an identical structure, those same states must also appear in the expansions of the logical states for $d \geqslant 3$. Therefore, the Hamiltonians derived here are sufficient to perform universal quantum computation using the three qudit DFS/NS.

Finally we note that our algorithm for the determination of compatible Hamiltonians is also applicable to DFS/NSs which are not collective. In addition, the generalized exchange interaction is compatible with any collective DFS/NS and can be analytically exponentiated to produce the corresponding unitary transformations, including SWAP gates, for qudits encoding qudits.

Appendix A. The algebra of $SU(d)$

We have chosen the following convention for the normalization of the algebra of Hermitian matrices which represent generators of $SU(d)$.

$$\text{tr}(\lambda_i \lambda_j) = 2\delta_{ij}. \tag{A.1}$$

The commutation and anti-commutation relations of the matrices representing the basis for the Lie algebra can be summarized by the following equation:

$$\lambda_i \lambda_j = \frac{2}{d}\delta_{ij} + \mathrm{i}f_{ijk}\lambda_k + d_{ijk}\lambda_k, \tag{A.2}$$

where here, and throughout this appendix, a sum over repeated indices is understood.

As with any Lie algebra we have the Jacobi identity:

$$f_{ilm}f_{jkl} + f_{jlm}f_{kil} + f_{klm}f_{ijl} = 0. \tag{A.3}$$

There is also a Jacobi-like identity,

$$f_{ilm}d_{jkl} + f_{jlm}d_{kil} + f_{klm}d_{ijl} = 0, \tag{A.4}$$

which was given by Macfarlane *et al* [64].

The following identities, also provided in [64], are useful in the derivation of the commutation relation given by equation (45),

$$d_{iik} = 0, \tag{A.5}$$

$$d_{ijk}f_{ljk} = 0, \tag{A.6}$$

$$f_{ijk}f_{ljk} = d\delta_{il}, \tag{A.7}$$

$$d_{ijk}d_{ljk} = \frac{d^2 - 4}{d}\delta_{il}, \tag{A.8}$$

and

$$f_{ijm}f_{klm} = \frac{2}{d}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (d_{ikm}d_{jlm} - d_{jkm}d_{ilm}) \tag{A.9}$$

and finally

$$f_{piq}f_{qjr}f_{rkp} = -\left(\frac{d}{2}\right)f_{ijk}, \tag{A.10}$$

$$d_{piq}f_{qjr}f_{rkp} = -\left(\frac{d}{2}\right)d_{ijk}, \tag{A.11}$$

$$d_{piq}d_{qjr}f_{rkp} = \left(\frac{d^2 - 4}{2d}\right)f_{ijk}, \tag{A.12}$$

$$d_{piq}d_{qjr}d_{rkp} = \left(\frac{d^2 - 12}{2d}\right)d_{ijk}. \tag{A.13}$$

The proofs of these are fairly straight-forward and are omitted.

Appendix B. Analytic expressions for unitary transformations

It was stated in 5.4 that the sum of all terms having the form $\lambda_i \otimes \lambda_i$ can be shown to commute with the sum of all terms $\lambda_j \otimes \lambda_j$, where the λ_i (λ_j) represent the diagonal (off-diagonal) basis elements of $SU(d)$. This appendix provides proof of this statement and gives an alternative description of the time evolution of the sum of off-diagonals.

B.1. Sum of diagonals and sum of off-diagonals commute

The objective of this appendix is to show that

$$\left[\sum_i \mu_{ii0}, \sum_j \mu_{jj0} \right] = 0, \tag{B.1}$$

where all λ_i are diagonal, all λ_j are off-diagonal and $\mu_{ijk} = \lambda_i \otimes \lambda_j \otimes \lambda_k$. It will be convenient to let $i \in I$, where I represents a subset of all numbers in the set $1, 2, \ldots, d^2 - 1$ which correspond to matrices λ_i which are diagonal and similarly $j \in J$, with J corresponding to indices with λ_j being off-diagonal matrices.

First we define

$$C_{od} = \sum_{ij} [\lambda_i \otimes \lambda_i, \lambda_j \otimes \lambda_j], \tag{B.2}$$

and then proceed to show that $C_{od} = 0$. Here again the convention is that repeated indices are to be summed. Using the identities provided in appendix A we may rewrite this last equation as

$$C_{od} = -2 f_{ijk} f_{ijl} \lambda_k \otimes \lambda_l + 2\mathrm{i} f_{ijk} d_{ijl} \lambda_k \otimes \lambda_l + 2 f_{ijk} f_{ijl} \lambda_l \otimes \lambda_k - 2\mathrm{i} f_{ijk} d_{ijl} \lambda_l \otimes \lambda_k$$
$$= 2\mathrm{i}(f_{ijk} d_{ijl} - f_{ijl} d_{ijk}) \lambda_k \otimes \lambda_l, \tag{B.3}$$

where the first equality follows from equation (A.2) and the second from relabeling. Now, since $f_{imk} = 0$ for both indices $i, m \in I$ we may establish the following equality:

$$\sum_{i,j} f_{ijk} d_{ijl} = \sum_{i,m} f_{imk} d_{iml}, \tag{B.4}$$

where $m = 1, 2, \ldots, d^2 - 1$. Since the index i now serves to distinguish the diagonal elements, the Jacobi-like identity, equation (A.4), can now be expressed as

$$2 f_{imk} d_{iml} + f_{lmk} d_{iim} = 0. \tag{B.5}$$

This implies that if

$$\sum_i d_{iil} = 0, \tag{B.6}$$

then $C_{od} = 0$ and the result follows. (Note that this is certainly true when i takes all values from 1 to $d^2 - 1$. Here, the i are in I so it may not be obvious.) To show this equation is satisfied for all d, we first state that it is true by direct computation for the matrix representation of the Lie algebra of $SU(3)$ using the Gell–Mann basis and also for $SU(4)$ using the Gell–Mann-like basis for $SU(4)$. (It is trivially true for $SU(1)$ and $SU(2)$ since all $d_{ijk} = 0$.) The only nonzero d_{iil} for $i \in I$ for $SU(3)$ are

$$d_{338} = 1/\sqrt{3} \quad \text{and} \quad d_{888} = -1/\sqrt{3},$$

and for $SU(4)$ they are

$$d_{3,3,8} = 1/\sqrt{3}, \qquad d_{8,8,8} = -1/\sqrt{3}, \qquad d_{3,3,15} = 1/\sqrt{6},$$
$$d_{8,8,15} = 1/\sqrt{6}, \qquad d_{15,15,15} = -2/\sqrt{6}. \tag{B.7}$$

Clearly, in both cases, $\sum_i d_{iil} = 0$. We now proceed by induction, showing equation (B.6) to be true for d while assuming it is valid for $d - 1$. We also note that this assumption and proof is motivated quite well by the d_{iik} for $SU(3)$ and $SU(4)$.

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By convention, we take a Gell–Mann basis for $SU(d)$ which has its diagonal elements of the form

$$\lambda_{d^2-1} = \sqrt{\frac{2}{d(d-1)}} \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & & & -(d-1) \end{pmatrix}, \tag{B.8}$$

where there are $d-1$ ones on the diagonal so that the matrix is traceless and normalized such that $\mathrm{Tr}(\lambda_{d^2-1})^2 = 2$. All diagonal matrices have this form and zeros are appended for higher dimensions. We now note that for all diagonal matrices equation (A.2) implies that the result of anti-commutation can only produce diagonal matrices. In other words, the λ_k in equation (A.2) are diagonal whenever the left-hand side of that equation represents the product of diagonal matrices. Now, noting that

$$\{\lambda_i, \lambda_{d^2-1}\} = 2\sqrt{\frac{2}{d(d-1)}}\lambda_i,$$

for $i \neq d^2 - 1$, we find that $d_{i,i,d^2-1} = \sqrt{2/d(d-1)}$ for all diagonal matrices λ_i which are confined to the upper $(d-1) \times (d-1)$ block. There are $d-2$ such matrices since there are a total of $d-1$ diagonal matrices. The only thing left to find is the following $d_{d^2-1,d^2-1,i}$. Let us calculate directly using equations (B.8) and (A.2),

$$\{\lambda_{d^2-1}, \lambda_{d^2-1}\} = \frac{4}{d(d-1)} \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & (d-1)^2 \end{pmatrix} = \frac{4}{d}\mathbb{1} + \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & & b \end{pmatrix},$$

where $a = 4(2-d)/d(d-1)$ and $b = 4(d-2)/d$. This implies that the only nonzero $d_{d^2-1,d^2-1,i}$ is

$$d_{d^2-1,d^2-1,d^2-1} = -(d-2)\sqrt{\frac{2}{d(d-1)}},$$

and thus $\sum_i d_{iij} = 0$ for $SU(d)$ assuming it is true for $SU(d-1)$.

B.2. Exponential of the sum of off-diagonals

Here we show how to calculate the exponential of $\sum_j \lambda_j \otimes \lambda_j$, we will first show that the square of $K = \frac{1}{2}\sum_{j\in J} \lambda_j \otimes \lambda_j$ is diagonal. (It is not proportional to the identity, but it is diagonal with the same entry for each non-zero element on the diagonal.)

To show this, we write the off-diagonal elements in one of the following forms,

$$|k\rangle\langle l| + |l\rangle\langle k| \qquad \text{or} \qquad i|k\rangle\langle l| - i|l\rangle\langle k|. \tag{B.9}$$

Note that $K = \sum_{k<l} Q_{k,l}$ so that K becomes

$$K = \frac{1}{2}\sum_{k<l}[(|k\rangle\langle l| + |l\rangle\langle k|) \otimes (|k\rangle\langle l| + |l\rangle\langle k|) + (i|k\rangle\langle l| - i|l\rangle\langle k|) \otimes (i|k\rangle\langle l| - i|l\rangle\langle k|)]$$

$$= \frac{1}{2}\sum_{k<l}[(|k\rangle\langle l| + |l\rangle\langle k|) \otimes (|k\rangle\langle l| + |l\rangle\langle k|) - (|k\rangle\langle l| - |l\rangle\langle k|) \otimes (|k\rangle\langle l| - |l\rangle\langle k|)]$$

$$= \sum_{k<l}[|k\rangle\langle l| \otimes |l\rangle\langle k| + |l\rangle\langle k| \otimes |k\rangle\langle l|]. \tag{B.10}$$

To square this, we calculate

$$K^2 = \sum_{p<q,k<l} [|p\rangle\langle q| \otimes |q\rangle\langle p| + |q\rangle\langle p| \otimes |p\rangle\langle q|][|k\rangle\langle l| \otimes |l\rangle\langle k| + |l\rangle\langle k| \otimes |k\rangle\langle l|]$$

$$= \sum_{p<q,k<l} [\delta_{qk}\delta_{pl}|p\rangle\langle l| \otimes |q\rangle\langle k| + \delta_{ql}\delta_{pk}|p\rangle\langle k| \otimes |q\rangle\langle l|]$$

$$+ \delta_{ql}\delta_{pk}|q\rangle\langle l| \otimes |p\rangle\langle k|] + \delta_{pl}\delta_{qk}|q\rangle\langle k| \otimes |p\rangle\langle l|].$$

However, note that the first term requires $p < q = k < l$ and $p = l$. But $p = l$ is impossible since p is strictly less than l. Therefore this term must vanish. Similarly for the last term. This gives

$$K^2 = \sum_{k<l} [|k\rangle\langle k| \otimes |l\rangle\langle l| + |l\rangle\langle l| \otimes |k\rangle\langle k|)], \tag{B.11}$$

which is clearly diagonal.

In fact, the form is clear. This matrix has ones everywhere on the diagonal except at certain places. These places are when the indices coincide, i.e., when $k = l$, since we are restricted to have k strictly less than l. For example, using the Gell–Mann basis for $SU(3)$, a straight-forward calculation using this formula produces $K^2 = \text{diag}\{0, 1, 1, 1, 0, 1, 1, 1, 0\}$.

It is now readily shown that

$$K^3 = K, \qquad \text{which implies} \quad K^4 = K^2, \qquad \text{etc.} \tag{B.12}$$

This allows us to analytically calculate the exponential of K and thus e_i which differs from K by the addition of diagonal terms (see equation (83)) and tensor products of the identity,

$$\exp[-itK] = (\mathbb{1} - K^2) + K^2 \cos(t) - iK \sin(t), \tag{B.13}$$

and thus we obtain SWAP by a slightly different method.

B.3. Exponential of the logical X operator

The objective is to calculate $\exp[-i\alpha \bar{X}]$ and $\exp[-i\alpha \bar{Z}]$ analytically. There are several ways that we could do this. Here we choose what we believe is the most straightforward.

Let us recall the definitions

$$\bar{X} = \sqrt{\frac{1}{12}}(e_1 - e_2) \tag{B.14}$$

and

$$\bar{Z} = \frac{1}{6}(e_1 + e_2 - 2e_3), \tag{B.15}$$

where

$$e_1 = \sum_i \lambda_0 \otimes \lambda_i \otimes \lambda_i, \tag{B.16}$$

$$e_2 = \sum_i \lambda_i \otimes \lambda_0 \otimes \lambda_i \tag{B.17}$$

and

$$e_3 = \sum_i \lambda_i \otimes \lambda_i \otimes \lambda_0. \tag{B.18}$$

The dimensionality of $\mathbb{1}$ should be clear from context and we will use $\mathbb{1}$ both for a d-state system, $\mathbb{1}_d$ and also the identity of a composite system $\mathbb{1}_d \otimes \mathbb{1}_d \otimes \cdots \otimes \mathbb{1}_d$.

The following identities may be shown, using the identities in appendix A,

$$e_i^2 = \frac{4}{d^2}(d^2 - 1)\mathbb{1} - \frac{4}{d}e_i. \tag{B.19}$$

Products of two have a cyclic property:

$$
\begin{aligned}
e_1 e_2 &= \frac{2}{d}\sum_i \lambda_i \otimes \lambda_i \otimes \mathbb{1} + \mathrm{i}\sum_{ijk} f_{ijk}\lambda_j \otimes \lambda_i \otimes \lambda_k + \sum_{ijk} d_{ijk}\lambda_j \otimes \lambda_i \otimes \lambda_k \\
&= \frac{2}{d}e_3 - \mathrm{i}F + D,
\end{aligned} \tag{B.20}
$$

and

$$e_1 e_3 = \frac{2}{d}e_2 + \mathrm{i}F + D, \tag{B.21}$$

$$e_2 e_3 = \frac{2}{d}e_1 - \mathrm{i}F + D. \tag{B.22}$$

Finally, we note that

$$e_i D = \left(\frac{4}{d^2}\right)\left(\frac{d^2 - 4}{d}\right)(e_j + e_k) - \frac{12}{d^2}D, \tag{B.23}$$

where $i = 1, 2, 3$ and $i \neq j \neq k \neq i$.

At this point the calculation proceeds in a straightforward albeit tedious manner. One simply computes $\bar{X}^3 = [(\sqrt{1/12})(e_1 - e_2)]^3$ and $\bar{Z}^3 = [(1/6)(e_1 + e_2 - 2e_3)]^3$ using the identities in this appendix as well as those in appendix A. After showing $\bar{Z}^3 = \bar{Z}$, we know that $\bar{Z}^4 = \bar{Z}^2$ (and similarly for $\bar{X}$) so the series may be summed to obtain the desired analytic expressions for the associated unitary transformations.

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